SOME ESTIMATORS FOR THE RATIO AND PRODUCT OF POPULATION PARAMETERS

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(Received : March, 1980)

SUMMARY

The use of ratio and product estimators employing information on an auxiliary variable is well known. Singh [2] considered the question of improving these estimators when supplementary information is available on auxiliary variable. In this paper alternative estimators have been proposed which have more practical utility.

Introduction

For any sample design, let $\hat{Y}$, $\hat{X}_1$ be the unbiased estimators of the population totals Y, X_1 of the variables y, x_1 respectively, and $\hat{X}_2$ be the unbiased estimator of the population total X_2 of the auxiliary variable x_2. Also, let C_0, C_1 and C_2 be the coefficients of variation of $\hat{Y}$, $\hat{X}_1$ and $\hat{X}_2$ and ρ_{01}, ρ_{02} and ρ_{12} be the correlation coefficients between $(\hat{Y}, \hat{X}_1)$, $(\hat{Y}, \hat{X}_2)$ and $(\hat{X}_1, \hat{X}_2)$ respectively.

For estimating the ratio R ($= Y/X_1$) the usual estimate is given by $\hat{R}$ $(= \hat{Y}/\hat{X}_1)$, Singh [2] proposed the estimators

$$\hat{R}_1^* = \hat{R}\,(\hat{X}_2/X_2) \text{ and}$$

$$\hat{R}^* = \hat{R}\,(X_2/\hat{X}_2)$$

which are, respectively, more efficient than $\hat{R}$ if

$$A < -1, \text{ and} \tag{1.1}$$

$$A > +1 \tag{1.2}$$

where $A = 2\,[\rho_{02}(C_0/C_2) - \rho_{12}(C_1/C_2)]$.

Similarly, for estimating the product $P(= YX_1)$ the usual estimator is given by $\hat{P}(= \hat{Y}\hat{X}_1)$ whereas Singh (1965) proposed the estimators

$$\hat{P}_1^* = \hat{P}(\hat{X}_2/X_2) \text{ and}$$

$$\hat{P}_2^* = \hat{P}(X_2/\hat{X}_2)$$

which are, respectively, more efficient than $\hat{P}$ if

$B < -1$ and

$B > 1$

where $B = 2\,[\rho_{02}(C_0/C_2) + \rho_{12}(C_1/C_2)]$.

In practical situations, however, it may happen that

$A < A_0(\neq -1)$ or $A > A_1(\neq 1)$ and

$B < B_0(\neq -1)$ or $B > B_1(\neq 1)$

We propose ratio-type and product-type estimators $\hat{R}_\theta$ and $\hat{P}_\theta$ which, for proper choice of the scalar constant θ depending on the value A_0, A_1, B_0 or B_1, may be made more efficient than the other estimators.

2. $\hat{R}_\theta$ and its Comparison with $\hat{R}$, $\hat{R}_1^*$ and $\hat{R}_2^*$

The proposed estimator $\hat{R}_\theta$ is defined as follows :

$$\hat{R}_\theta = (1 + \theta)\,\hat{R} - \theta\,\hat{R}_1^*$$

Let $\hat{Y} = Y(1 + e_0)$, $X_1 = X_1(1 + e_1)$ and $X_2 = X_2(1 + e_2)$ where it is assumed that the sample is large enough to make $|\,e_i\,|$, $i = 0, 1, 2$ so small that terms of degree greater than two in e_i's may be negligible to justify the first degree approximation to the mean square error (MSE) of the estimators.

We may easily check that, to the first degree of approximation

$$\text{MSE}(\hat{R}_\theta) = \text{MSE}(\hat{R}) + R^2C_2^2\theta^2\,(1 - A/\theta) \tag{2.1}$$

It is known that, to the first degree of approximation

$$\text{MSE}(\hat{R}_1^*) = \text{MSE}(\hat{R}) + R^2C_2^2\,(1 + A) \tag{2.2}$$

$$\text{MSE}(\hat{R}_2^*) = \text{MSE}(\hat{R}) + R^2C_2^2\,(1 - A) \tag{2.3}$$

It follows from (2.1) that $\text{MSE}(\hat{R}_\theta) < \text{MSE}(\hat{R})$ if

$$R^2C_2^2\theta^2\,(1 - A/\theta) < 0$$

that is, if either $A < \theta < 0$ (2.4)

or $0 < \theta < A$ (2.5)

For example, if it is known that $A < A_0$ (< 0) we may choose $\theta = A_0$ and if $A > A_1$ (> 0) we may choose $\theta = A_1$ to make $\hat{R}_\theta$ more efficient than $\hat{R}$ in both the cases. Therefore, even when A lies in range $(-1, 1)$ we get estimators $\hat{R}_\theta$ which are better than $\hat{R}$ while, according to Singh [2], this may not be true about $\hat{R}_1^*$ and $\hat{R}_2^*$.

Again, it follows from (2.1) and (2.2) that

$$\text{MSE}(\hat{R}_\theta) < \text{MSE}(\hat{R}_1^*)$$

if $\theta^2(1 - A/\theta) < (1 + A)$

or if $(\theta + 1)(\theta - 1 - A) < 0$

that is, if either $A + 1 < \theta < -1$ (2.6)

or $-1 < \theta < A + 1$ (2.7)

For example, if it is known that $A < A_0$ (< -2) we may choose $\theta = A_0 + 1$ and if $A > A_1$ (> -2) we may choose $\theta = A_1 + 1$ to make $\hat{R}_\theta$ more efficient that $\hat{R}_1^*$ in both the cases.

Similarly, it follows from (2.1) and (2.3) that

$$\text{MSE}(\hat{R}_\theta) < \text{MSE}(\hat{R}_2^*)$$

if either $A - 1 < \theta < 1$ (2.8)

or $1 < \theta < A - 1$ (2.9)

For example, if it is known that $A < A_0$ (< 2) we may choose $\theta = A_0 - 1$ and if $A > A_1$ (> 2) we may choose $\theta = A_1 - 1$ to make $\hat{R}_\theta$ more efficient than $\hat{R}_2^*$ in both the cases.

3. Optimum θ and its Estimate

The optimum θ for which MSE($\hat{R}_\theta$) is minimised is given by

$$\theta_{opt.} = \rho_{02}(C_0/C_2) - \rho_{12}(C_1/C_2) = \frac{A}{2} = C.$$

For this value of $\theta_{opt.}$, the minimum mean square error is given by

$$\text{MSE}(\hat{R}_\theta)_{min.} = \text{MSE}(R) - R^2 C_2^2 C^2 \tag{3.1}$$

In practice if a good guess-value of θ is available on the basis of past data, pilot study or experience, this information can be utilised to get the estimator $\hat{R}_{\theta_{opt.}}$ with its minimum mean square error given by (3.1).

Exact value of $\theta_{opt.}$ may be rarely known in practice, hence it is advisable to estimate $\theta_{opt.}$ from sample values. We can write

$$\rho_{02}(C_0/C_2) = \frac{S_{02}}{S_2^2}\left(\frac{X_2}{Y}\right)$$

$$\rho_{12}(C_1/C_2) = \frac{S_{12}}{S_2^2}\left(\frac{X_2}{X_1}\right)$$

where S_{02}, S_{12} and S_2^2 are cov. $(\hat{Y}, \hat{X}_2)$, cov. $(\hat{X}_1, \hat{X}_2)$ and $V(\hat{X}_2)$ respectively. Let us denote s_{02}, s_{12} and s_2^2 be the unbiased estimators of S_{02}, S_{12} and S_2^2 respectively, so that $\theta_{opt.}$ is estimated by

$$\hat{C} = \left(\frac{s_{02}}{s_2^2}\right)\frac{\hat{X}_2}{\hat{Y}} - \left(\frac{s_{12}}{s_2^2}\right)\frac{\hat{X}_2}{\hat{X}_1}$$

which is when substituted in $\hat{R}_\theta$ inplace of θ, we get the resulting estimator as

$$\hat{R}_{\hat{\theta}_{opt.}} = (1 + \hat{C})\,\hat{R} - \hat{C}\hat{R}_1^* \tag{3.2}$$

Nor we find the mean square error of $\hat{R}_{\hat{\theta}_{opt.}}$.

Let us define

$$s_{02} = S_{02}\,(1 + e_3),\ s_{12} = S_{12}\,(1 + e_4),\ s_2^2 = S_2^2\,(1 + e_5)$$

so that $E(e_3) = E(e_4) = E(e_5) = 0$,

We have

$$\hat{R}_{\hat{\theta}_{opt.}} = (1 + \hat{C})\,\hat{R} - \hat{C}\hat{R}\left(\frac{\hat{X}_2}{X_2}\right)$$

$$= \hat{R}\left\{ 1 - \hat{C}\left(\frac{\hat{X}_2}{X_2}\right) - 1 \right\}$$

$$= \frac{\hat{Y}}{\hat{X}_1}\left[1 - \left\{\frac{s_{02}}{s_2^2}\frac{\hat{X}_2}{\hat{Y}} - \frac{s_{12}}{s_2^2}\frac{\hat{X}_2}{\hat{X}_1}\right\}\left(\frac{\hat{X}_2}{X_2} - 1\right)\right]$$

$$= \frac{Y(1 + e_0)\,(1 + e_1)^{-1}}{X_1}\left[1 - \left\{\frac{S_{02}(1 + e_3)}{Y(1 + e_0)} - \frac{S_{12}(1 + e_4)}{X_1(1 + e_1)}\right\} \cdot \frac{X_2(1 + e_2)}{S_2^2(1 + e_5)} \cdot e_2\right]$$

$$= R\,[(1 + e_0)\,(1 - e_1 + e_1^2 - \ldots) - (1 + e_0)\,(1 + e_1)^{-1} \cdot \{\rho_{02}(C_0/C_2) \cdot (1 + e_3)\,(1 + e_0)^{-1} - \rho_{12}(C_1/C_2)\,(1 + e_4)\,(1 + e_1)^{-1}\}\,(1 + e_2)\,(1 + e_5)^{-1}\,e_2]$$

or $\hat{R}_{\hat{\theta}_{opt.}} - R$

$$= R\,[(e_0 - e_1 + e_1^2 - \ldots) - (1 + e_0 - e_1 + e_1^2 - \ldots)\,\{\rho_{02}(C_0/C_2)\,(1 + e_3)\,(1 + e_0)^{-1} - \rho_{12}(C_1/C_2)\,(1 + e_4)\,(1 + e_1)^{-1}\}\,(1 + e_2)\,(1 + e_5)^{-1}\,e_2] \qquad (3.3)$$

Squaring both sides, taking expectation we have to the first degree of approximation.

$$\text{MSE}(\hat{R}_{\hat{\theta}_{opt.}}) = R^2E\,[(e_0 - e_1) - \{\rho_{02}(C_0/C_2) - \rho_{12}(C_1/C_2)\}\,e_2]^2$$

$$= R^2\,[E(e_0 - e_1)^2 + C^2E(e_2^2) - 2C\,E(e_0e_2 - e_1e_2)]$$

$$= \text{MSE}(\hat{R}) + R^2\,[C^2C_2^2 - 2C(\rho_{02}C_0C_2 - \rho_{12}C_1C_2)]$$

$$= \text{MSE}(\hat{R}) + R^2C_2^2\,[C^2 - 2C^2]$$

$$= \text{MSE}(\hat{R}) - R^2C_2^2C^2 \qquad (3.4)$$

$$= \text{MSE}(\hat{R}_{\theta_{opt.}}) \qquad (3.5)$$

From (3.5), we see that the estimator $\hat{R}_{\hat{\theta}_{opt.}}$ when θ is estimated from sample values, attains the minimum mean square error given by (3.4) or (3.1). Noting $A/2 = C$ and on comparison, from (2.2), (2.3) and (3.4) it is clear that the estimator $\hat{R}_{\hat{\theta}_{opt.}}$ is always more efficient than $\hat{R}$, $\hat{R}_1^*$ and $\hat{R}_2^*$.

4. Estimator $\hat{P}_\theta$ and its Comparison with $\hat{P}$, $\hat{P}_1^*$ and $\hat{P}_2^*$

The proposed estimator $\hat{P}_\theta$ is defined as follows :

$$\hat{P}_\theta = (1 + \theta)\,\hat{P} - \theta\hat{P}_1^* \tag{4.1}$$

The mean square error of $\hat{P}_\theta$ is

$$\text{MSE}(\hat{P}_\theta) = \text{MSE}(\hat{P}) + P^2C_2^2\theta^2\left(1 - \frac{B}{\theta}\right) \tag{4.2}$$

Also, from Singh [2]

$$\text{MSE}(\hat{P}_1^*) = \text{MSE}(\hat{P}) + P^2C_2^2\theta^2\,(1 + B) \tag{4.3}$$

$$\text{and } \text{MSE}(\hat{P}_2^*) = \text{MSE}(\hat{P}) + P^2C_2^2\theta^2\,(1 - B) \tag{4.4}$$

It follows from (4.2) that $\text{MSE}(\hat{P}_\theta) < \text{MSE}(\hat{P})$ if

$$P^2C_2^2\theta^2\,(1 - B/\theta) < 0$$

that is, if either $B < \theta < 0$

or $0 < \theta < B$

For example, if it is known that $B < B_0$ (<0) we may choose $\theta = B_0$ and if $B > B_1$ (>0) we may choose $\theta = B_1$ to make $\hat{P}_\theta$ more efficient than $\hat{P}$ in both the cases. Therefore, even when B lies in the range $(-1, 1)$ we get estimators $\hat{P}_\theta$ which are better than $\hat{P}$ while, according to Singh [2], this may not be true for $\hat{P}_1^*$ and $\hat{P}_2^*$.

Again, it follows from (4.2) and 4.3) that

$$\text{MSE}(\hat{P}_\theta) < \text{MSE}(\hat{P}_1^*)$$

if $\theta^2(1 - B/\theta) < (1 + B)$

or if $(\theta + 1)\,(\theta - 1 - B) < 0$

that is, if either $B + 1 < \theta < -1$

or $-1 < \theta < B + 1$

For example, if it is known that $B < B_0 (< -2)$ we may choose $\theta = B_0 + 1$ and if $B > B_1 (> -2)$ we may choose $\theta = B_1 + 1$ to make $\hat{P}_\theta$ more efficient than $\hat{P}_1^*$ in both the cases.

Similarly, it follows from (4.2) and (4.4) that

$$\text{MSE}(\hat{P}_\theta) < \text{MSE}(\hat{P}_2^*)$$

if either $B - 1 < \theta < 1$

or $1 < \theta < B - 1$

For the example, if it is known that $B < B_0 (< 2)$ we may choose $\theta = B_0 - 1$ and if $B > B_1 (>2)$ we may choose $\theta = B_1 - 1$ to make $\hat{P}_\theta$ more efficient than $\hat{P}_2^*$ in both the cases.

The optimum θ for which $\text{MSE}(\hat{P}_\theta)$ is minimum is given by $C^* = B/2$. For this optimum θ and its estimated value $\hat{C}^* = (s_{02}/s_2^2)\, \hat{X}_2/\hat{Y} + (s_{12}/s_2^2)\, \hat{X}_2/\hat{X}_1$, we get similar results as those in case of $\hat{R}_\theta$.

5. An Illustration

We consider the following example given by Singh. [2] :

The data for all 61 blocks of Ahmedabad city Ward No. 1 (Khadia I) taken from 1961 population census have been considered for the purpose of this study. It is intended to determine the proportion (R) of 'total females employed (Y)' to the 'total female population (X_1)'. The supplementary characteristic chosen for this purpose is the 'females in services (X_2) (group IX of population census).' For this population we have

$Y = 455$	$C_0'^2 = 0.5046$	$\rho_{01} = 0.0388$
$X_1 = 19198$	$C_1'^2 = 0.0379$	$\rho_{02} = 0.7737$
$X_2 = 324$	$C_2'^2 = 0.5737$	$\rho_{12} = -0.0474$

where $C_i^2 = kC_i'^2$, where $C_i'^2$ stands for squares of the coefficient of variation for the characteristics and k is a constant given by $N - n/(N-1)\, n$ where N and n are respectively the number of blocks in the population and sample drawn with equal probability without replacement.

For this example $A = 1.4755$ satisfying the condition (1.2) so that $\hat{R}_2^*$ is more efficient than $\hat{R}$ with

$$\text{MSE}(\hat{R}) = k'\,(0.5318) \quad \text{and}$$

$$\text{MSE}(\hat{R}_2^*) = k'\,(0.2542)$$

where $k' = kR^2$.

We have $A - 1 = .4755$. Since $A < 2$, $\hat{R}_\theta$ can be made more efficient than $\hat{R}_2^*$ by choosing $.4755 < \theta < 1$ satisfying the condition (2.8). The following table shows MSE($\hat{R}_\theta$) for different values of θ in the vicinity of optimum θ.

θ	MSE($\hat{R}_\theta$)
.5	K' (0.2520)
.6	K' (0.2304)
.74 (opt.)	K' (0.2195)
.8	K' (0.2218)
.9	K' (0.2346)
.95	K' (0.2453)

6. Double Sampling

Let n' units are selected in the first phase and n units in the second phase according to any specified sample design, and let $\hat{X}_2'$ be an unbiased estimator of X_2 based on the first phase sample and, $\hat{Y}$ $\hat{X}_1$, $\hat{X}_2$ be the unbiased estimators of Y, X_1, X_2 based on the second phase sample respectively.

The proposed double sampling estimators of R and P are

$$\hat{R}_{\theta d} = (1 + \theta)\,\hat{R} - \theta \hat{R}_{1d} \tag{6.1}$$

and $$\hat{P}_{\theta d} = (1 + \theta)\,\hat{P} - \theta \hat{P}_{1d} \tag{6.2}$$

respectively, where

$$\hat{R} = \frac{\hat{Y}}{\hat{X}_1}, \quad \hat{R}_{1d} = \hat{R}\left(\frac{\hat{X}_2}{\hat{X}_2'}\right)$$

and

$$\hat{P} = \hat{Y}\hat{X}_1, \quad \hat{P}_{1d} = \hat{P}\left(\frac{\hat{X}_2}{\hat{X}_2'}\right).$$

Substituting $\hat{Y} = Y(1 + e_1)$, $\hat{X}_1 = X_1(1 + e_1)$, $\hat{X}_2 = X_2(1 + e_2)$ and $\hat{X}_2' = X_2(1 + e_2')$ we have to the first degree of approximation

$$\text{MSE}(\hat{R}_{\theta d}) = R^2 E[(e_0 - e_1) - \theta(e_2 - e_2')]^2$$

$$= \text{MSE}(\hat{R}) + R^2\Big[\theta^2\Big\{\frac{V(\hat{X}_2)}{X_2^2} + \frac{V(\hat{X}_2')}{X_2^2} - \frac{2\text{Cov.}(\hat{X}_2, \hat{X}_2')}{X_2^2}\Big\}$$

$$- 2\theta\Big\{\frac{\text{Cov.}(\hat{Y}, \hat{X}_2)}{YX_2} - \frac{\text{Cov.}(\hat{X}_1, \hat{X}_2)}{X_1X_2} - \frac{\text{Cov.}(\hat{Y}, \hat{X}_2)}{YX_2}$$

$$+ \frac{\text{Cov.}(\hat{X}_1, \hat{X}_2')}{X_1X_2}\Big\}\Big] \tag{6.3}$$

It may be noted here that in case of simple random sampling without replacement at both the phases

$$\text{MSE}(\hat{R}_{\theta d}) = R^2\Big[\Big(\frac{1}{n} - \frac{1}{N}\Big)(C_1'^2 - 2\rho_{01}'C_0'C_1' + C_1'^2)$$

$$+ \theta^2\Big(\frac{1}{n} - \frac{1}{n'}\Big)C_2'^2 - 2\theta\Big(\frac{1}{n} - \frac{1}{n'}\Big)$$

$$\{\rho_{02}'C_0'C_2' - \rho_{12}'C_1'C_2'\}\Big] \tag{6.4}$$

where N: the population size, $f = N - n/N$, C_0', C_1', C_2' are coefficients of variations of the variables y, x_1, x_2 respectively, and ρ_{01}', ρ_{02}' and ρ_{12}' are correlation coefficients between (y, x_1), (y, x_2) and (x_1, x_2) respectively.

Putting $A' = 2\,[\rho_{02}'(C_0'/C_2') - \rho_{12}'(C_1'/C_2')]$, we have

$$\text{MSE}(\hat{R}_{\theta d}) = R^2\Big[\Big(\frac{1}{n} - \frac{1}{N}\Big)(C_0'^2 - 2\rho_{01}'C_0'C_1' + C_1'^2)$$

$$+ \Big(\frac{1}{n} - \frac{1}{n'}\Big)\theta^2 C_2'^2\Big\{1 - \frac{A'}{\theta}\Big\}\Big] \tag{6.5}$$

Also, the mean square errors of double sampling estimators suggested by Singh (1965) are

$$\text{MSE}(\hat{R}_{1d}) = R^2\Big[\Big(\frac{1}{n} - \frac{1}{N}(C_0'^2 - 2\rho_{01}'C_0'C_1' + C_1'^2)$$

$$+ \Big(\frac{1}{n} - \frac{1}{n'}\Big)C_2'^2(1 + A')\Big] \tag{6.6}$$

$$\text{MSE}(\hat{R}_{2d}) = R^2\Big[\Big(\frac{1}{n} - \frac{1}{N}\Big)(C_0'^2 - 2\rho_{01}'C_0'C_1' + C_1'^2)$$

$$+ \Big(\frac{1}{n} - \frac{1}{n'}\Big)C_2'^2(1 - A')\Big] \tag{6.7}$$

From (6.5) to (6.7), comparison among the double sampling estimators gives efficiency conditions similar to those found for single sampling.

Similar results for $\widehat{P}_{\Theta d}$ can be obtained.

ACKNOWLEDGEMENT

The authors wish to thank the referees for their useful suggestions.

REFERENCES

[1] Murthy, M. N. (1967) : *Sampling Theory and Methods*, Statistical Publishing Society.

[2] Singh, M. P. (1965) : On the estimation of ratio and product of the population parameters, *Sankhya*, 27(B); 321-328.